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Title: Towards an Analytical Model for Carbon Storage in Forested Landscapes

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Enrique A. Thomann

An analytical model is developed to address the question of how different disturbance regimes affect the mean and variance of landscape carbon storage in forest ecosystems. Total landscape carbon is divided into five pools based on the processes from which they are derived and based on their temporal dynamics. Formulae for the mean and variance of each pool and the total landscape carbon are obtained for the different disturbance regimes and their dependence of the statistical measures on two key parameters, the observation window and the mean disturbance interval, are analyzed. Comparison of the predictions of the analytical model with those from Maxcarb is included. The analytical solution consistently matches the prediction of Maxcarb. While the analytical model might not replace detailed disturbance simulation models such as Maxcarb, it can more expeditiously provide the qualitative trends of these models. The analytical model might be combined with the results of physiologically-based models (which generally exclude disturbance) to make adjustments for the presence of disturbances in forested landscapes.

Towards an Analytical Model for Carbon Storage in Forested Landscapes

by

Nam V. Ngo

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APPROVED:

Major Professor, representing Mathematics

Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Nam V. Ngo, Author

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I want to thank Zac Kayler and Liz Borrows who initially worked with me on this research project.

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CONTRIBUTION OF AUTHORS

Professor Enrique A. Thomann assisted in the mathematics during the formulation and derivation of the analytical model. Professor Mark E. Harmon initiated the topic of this thesis, helped me to formulate and understand the significance of the problem, and importantly helped me in the writing of this thesis. Dr. Erica A. H. Smithwick and Jimm Domingo developed the Maxcarb model which was used in the thesis. Adam Moreno assisted me in running the simulation Maxcarb model. Liz Borrows and Zac Kayler initially worked with me on the project.

TABLE OF CONTENTS

LIST OF TABLES

TOWARDS AN ANALYTICAL MODEL FOR CARBON STORAGE IN FORESTED LANDSCAPES

1. INTRODUCTION

Management of carbon storage in forested ecosystems is of key importance in policies that address global changes in climate related to greenhouse gases such as carbon dioxide (IPCC report, Kyoto Protocol). Forests have a great potential to sequester carbon and thus, they play an important role in removing carbon dioxide from the atmosphere (Harmon, 2001). Carbon sequestration describes the collective processes that remove carbon from the atmosphere. Even though many carbon sequestration processes occur at the molecular level (e.g., photosynthesis, formation, and protection of soil organic matter) management practices to enhance carbon sequestration will be implemented at the landscape level and therefore have to incorporate the effects of disturbance. Disturbances are known to significantly influence forest carbon storage (Janisch and Harmon 2002; Wirth et al. 2002; Bond-Lamberty; Law et al. 2004; Smithwick et al. in press). Understanding how disturbance regimes affect carbon storage in a forested ecosystem can help lead to the development of a viable carbon sequestration policy.

Examples of current models developed to predict changes in carbon storage in forested ecosystem are BIOME-GBC (Running et al., 1993) and CASA (Cramer et al., 1999). These models are primarily physiologically based and for the most part do not address effects of disturbance on landscape carbon storage. The MaxCarb model is one of the recent models developed to address potential carbon storage in forested ecosystems with the effect of disturbances (Smithwick et al. in press). These models and many other models that have developed to predict changes in carbon storage in forested ecosystem are simulation based and thus rely greatly on computers to make extensive computations that can take considerable time and resources. In addition, simulation models are complex in their relationship of variables and hence, results may be difficult to interpret.

In this thesis I take an alternative approach by developing a mathematical model to address how disturbance regimes can affect carbon storage in a forested landscape using a more limited set of parameters and initial conditions. I derive an analytical solution and use it to qualitatively explore how carbon storage is influenced under different disturbance regimes (regular, random, and no disturbance) and to test if the analytical solution can predict the general trends and relationships that have been predicted by simulation models. In particular, I use the analytical solution to examine how the mean and variance of landscape carbon storage vary over time for the case without catastrophic disturbance as well as for different mean disturbance intervals in regulated and random disturbance regimes. Finally, I quantitatively compare the results of our new mathematical model with the results of the MaxCarb model.

This research project grew out of one of the projects introduced from IGERT Ecosystems Informatics classes at Oregon State University during the academic years of 2004 and 2005. The classes were co-taught by Professor Julia A. Jones from department of Geosciences, Professor Mark E. Harmon from department of Forest Science, Professor Tom Dietterich from department of Computer Science, and Professor Enrique A. Thomann from department of Mathematics. The classes nurtured the interaction that is reflected in this project. The project was initiated by Professor Mark E. Harmon, who has extensively worked on landscape carbon dynamics using simulation models. In the spirit of integrating the tools and concepts from mathematics and computer science to ecosystems, we recognized the potential for a mathematical model which can be used in conjunction of the simulation models that can help to better understand this topic. As a result, the project was introduced and was initially explored by Zac Kayler, a student in Forest Science, Liz Borrows, a student in Bioengineering, and I myself, a student in Mathematics. Zac Kayler and Liz Borrows have been busy with their dissertation projects and thus, leaving myself with the help of Professor Mark E. Harmon and Professor Enrique A. Thomann to explore this project further.

2. MANUSCRIPT

TOWARDS AN ANALYTICAL MODEL FOR CARBON STORAGE IN FORESTED LANDSCAPES

Abstract

An analytical model is developed to address the question of how different disturbance regimes affect the mean and variance of landscape carbon storage in forest ecosystems. Total landscape carbon is divided into five pools based on the processes from which they are derived and based on their temporal dynamics. Formulae for the mean and variance of each pool and the total landscape carbon are obtained for the different disturbance regimes and their dependence of the statistical measures on two key parameters, the observation window and the mean disturbance interval, are analyzed. Comparison of the predictions of the analytical model with those from Maxcarb is included. The analytical solution consistently matches the prediction of Maxcarb. While the analytical model might not replace detailed disturbance simulation models such as Maxcarb, it can more expeditiously provide the qualitative trends of these models. The analytical model might be combined with the results of physiologically-based models (which generally exclude disturbance) to make adjustments for the presence of disturbances in forested landscapes.

2.1. Introduction

Management of carbon storage in forested ecosystems is of key importance in policies that address global changes in climate related to greenhouse gases such as carbon dioxide (IPCC report, Koyoto Protocol). Forests have a great potential to sequester carbon and thus, they play an important role in removing carbon dioxide from the atmosphere (Harmon, 2001). Carbon sequestration describes the collective processes that remove carbon from the atmosphere. Even though many carbon sequestration processes occur at the molecular level (e.g., photosynthesis, formation, and protection of soil organic matter) management practices to enhance carbon sequestration will be implemented at the landscape level and therefore have to incorporate the effects of disturbance. Disturbances are known to significantly influence forest carbon storage (Janisch and Harmon 2002; Wirth et al. 2002; Bond-Lamberty; Law et al. 2004; Smithwick et al. in press). Understanding how disturbance regimes affect carbon storage in a forested ecosystem can help lead to the development of a viable carbon sequestration policy.

Examples of current models developed to predict changes in carbon storage in forested ecosystem are BIOME-GBC (Running et al., 1993) and CASA (Cramer et al., 1999). These models are primarily physiologically based and for the most part do not address effects of disturbance on landscape carbon storage. The MaxCarb model is one of the recent models developed to address potential carbon storage in forested ecosystems with the effect of disturbances (Smithwick et al. in press). These models and many other models that have developed to predict changes in carbon storage in forested ecosystem are simulation based and thus rely greatly on computers to make extensive computations that can take considerable time and resources. In addition, simulation models are complex in their relationship of variables and hence, results may be difficult to interpret.

In this paper we take an alternative approach by developing a mathematical model to address how disturbance regimes can affect carbon storage in a forested landscape using a more limited set of parameters and initial conditions. We derive the analytical solution and use it to qualitatively explore how carbon storage is influenced under different disturbance regimes and to test if the analytical solution can predict the general trends and relationships that have been predicted by simulation models. In particular, we use the analytical solution to examine how the mean and variance of landscape carbon storage vary over time for the case without catastrophic disturbance as well as for different mean disturbance intervals in regulated and random disturbance regimes. Finally, we quantitatively compare the results of our new mathematical model with the results of the MaxCarb model.

2.2. General Concepts

Disturbances are known to influence carbon storage in forest (Janisch and Harmon 2002; Wirth et al. 2002; Bond-Lamberty; Law et al. 2004; Smithwick et al. in press). Some common disturbances influencing forest carbon are harvest, clearing, fire, wind, insects, disease, flood, earth movements, and volcanism. Immediately following a major disturbance, decomposing material created by the disturbance typically loses more carbon than the re-growing forest gains (Smithwick et al. in press). As the re-growing stand matures, it begins to take in more carbon than it releases via decomposition and the stand begins to accumulate carbon. If another major disturbance does not occur for a long period of time, then carbon uptake by the re-growing forest eventually equals that lost from decomposing material and the ecosystem is in net balance regarding carbon fluxes (Harmon, 2001).

We define disturbances as discrete events that lead to a restructuring of age classes of stands in the landscape. Disturbance regimes, which are comprised of a population of disturbance events, can be characterized in part by the average disturbance interval, the regularity of disturbance events (i.e., regular or random), and the severity of disturbance events (i.e., amount of carbon removed). A disturbance regime creates a landscape age-class structure (Smithwick et al. 2005). Disturbances cause transfers from live pools to dead pools. Moreover, in cases of fires and harvest there are transfers to the atmosphere and forest product pools, respectively.

In our model, we consider a forested landscape consisting of multiple stands. Each stand is part of an age-class created by a concurrent set of disturbances. Hence, many stands can be in the same age-class. Considering all the stands in a landscape, the age class structure can be described by a frequency of distribution. The carbon stored in the landscape is the sum of all carbon pools in all stands. One can categorize carbon pools in a stand in many different ways. For our purpose, we divide total carbon in the stand into pools based on the processes from which they are derived and based on their temporal dynamics. This leads to five pools: 1) a live accumulating pool, 2) a dead accumulating pool, 3) a dead decreasing pool, 4) a forest product pool, and 5) a stable pool. The sizes of these pools are functions of age and are quantified in units of carbon stores per unit area. The live accumulating pool generally increases over time and is derived from plant production (i.e., photosynthesis). The dead accumulating pool is derived from the death of the live accumulating pool. Although once a tree dies it begins to decompose and thus lose carbon, if trees die at a steady rate, the mass of carbon stored in this dead tree pool can accumulate (Harmon, 2001). The dead decreasing pool is derived from the live pool killed by the disturbance and a part of the dead pool that was left after the disturbance. Given that this input is very large and infrequent; this pool tends to increase markedly after disturbance and then decrease as decomposition proceeds. The forest product pool is derived from harvest of the live and dead accumulating pools and used by humans. This pool also increases after disturbance depending on the amount harvested and the efficiency of humans in converting harvested material into long term forest products. This pool also tends to decrease over time until the next harvest. The stable pool is derived from very decomposed dead material. Given that it has relatively constant inputs from the dead pools and low rate of losses, this pool does not change greatly over time and can be approximated as a steady-state store. It has relatively constant inputs from the dead pools and low rates of losses.

2.3. Mathematical Framework

2.3.1 Model Formulation

Figure 1 illustrates how each carbon pool in a stand responds to disturbances through time. We assume all stands in a landscape have the same pools, that is a live accumulating pool, a dead accumulating pool, a dead decreasing pool, a forest product pool, and a stable pool. Parameters in each pool can be different from stand to stand. For example, each stand can have different initial age, accumulation rate, or decomposition rate. We consider a landscape consisting of M stands, labeled from 1 through M. First we describe the formulation of how the m^{th} stand sequesters carbon as a function of time.

Let

$$\{\tau_j^{(m)}\}_{j=1}^{\infty}, \quad \text{for } 1 \leq m \leq M,$$

denote the sequence of the inter-arrival times of disturbances in the m^{th} stand. The regularity of disturbances (i.e. regular or random) is characterized by the regularity of this sequence.

Let $t_j^{(m)}$ denote the arrival times of the j^{th} disturbance in the m^{th} stand. Then,

$$t_j^{(m)} = \sum_{i=1}^{j} \tau_i^{(m)}, \quad \text{for all } j \geq 1,$$

and define the initial time,

$$t_0^{(m)} = 0.$$

Note that the length of time between successive disturbance is

$$\tau_j^{(m)} = t_j^{(m)} - t_{j-1}^{(m)}, \quad \text{for all } j \geq 1.$$

Let

$$N_m(t) = min\{k : \tau_1^{(m)} + \tau_2^{(m)} + ... + \tau_k^{(m)} \leq t < \tau_1^{(m)} + \tau_2^{(m)} + ... + \tau_k^{(m)} + \tau_{k+1}^{(m)}\}$$

denote the number of disturbances and $t_{N_m(t)}$ denote the time of the last disturbance in the m^{th} stand by time t.

Let $a_m(0)$ denote the initial age of the m^{th} stand. Then, our assumptions lead to the following model for the age of the m^{th} stand at time t,

$$a_m(t) = \begin{cases} a_m(0) + t & \text{if } t_1^{(m)} > t \\ t - t_{N^{(m)}(t)} & \text{if } t \geq t_1^{(m)}. \end{cases} \tag{2.1}$$

Using this age function, we model carbon pools in the m^{th} stand as follows:

The live accumulating and dead accumulating pools are modeled by functions of age. Examples that are frequently used to model these pools are Chapman-Richard functions, that is,

$$f(s) = A(1 - e^{-as})^q, \tag{2.2}$$

and

$$g(s) = \alpha A(1 - e^{-bs})^p. \tag{2.3}$$

Here A is the maximum amount of carbon in the live accumulating pool, a is the live accumulation rate, q is the live accumulation lag, $0 \leq \alpha \leq 1$, is the dead rate from the accumulating live carbon, b is the dead accumulation rate, and p is the dead accumulation lag.

Each time a disturbance event occurs portions of the live and dead accumulating pools get transferred into the dead decreasing pool and the forest product pool where it is assumed they decrease exponentially over time at different rates. The amount of carbon stored in each of these pools at any given time is a function of the times at which the disturbance happened and the elapsed times since those disturbances have occurred. For each stand the dead decreasing pool, denoted by $d(t)$, and the forest product pool, denoted by $p(t)$, can be modeled by the stochastic processes

$$d(t) = \sum_{j=1}^{N(t)} \mu[f(\tau_j) + g(\tau_j)]e^{-c(t-t_j)}, \tag{2.4}$$

and

$$p(t) = \sum_{j=1}^{N(t)} \beta[f(\tau_j) + g(\tau_j)]e^{-d(t-t_j)}. \tag{2.5}$$

Here, c is the decomposition rate, μ is the fraction of live and dead pools that is not removed by the disturbance, d is the decomposition rate, and β is the fraction of the live and dead pools that is harvested and converted into long term products.

Unlike the live accumulating, dead accumulating, dead decreasing, and forest product pools, the stable pool is a function of the average of dead accumulating and dead decreasing carbon with a proportion forming stable material as defined by the stable formation rate η, and losses are controlled by the stable decomposition rate γ . In each stand the stable pool, denoted by $s(t)$, is modeled as follows:

Average Total Dead = Average of dead accumulating pool + Average of dead decreasing pool

Input Stable = (stable formation rate)*(Average Total Dead)

$$s(t) = \frac{\text{Input Stable}}{\text{stable decomposition rate}}. \tag{2.6}$$

Finally, let $c_m(t)$ denote the total carbon sequestered by the m^{th} stand at time t. It is the sum of the five pools above. We therefore can specify this as:

$$c_m(t) = f(a_m(t)) + g(a_m(t)) + d_m(t) + p_m(t) + s_m(t). \tag{2.7}$$

The analysis of the carbon at the stand level requires the study of the statistical properties of a stochastic process of the form

$$c(t) = h(a(t)) + \sum_{j=1}^{N(t)} A(\tau_j)e^{-r(t-t_j)}, \quad t \geq 0.$$

This analysis will be done in the following subsections by deriving the mean and variance of each carbon pool and the total landscape carbon at the stand and landscape level.

We assume that the landscape has area $|L|$, and is partitioned into M stands. Let $|l_m|$ denote the area of the m^{th} stand which we assume remains constant over time. Hence, for a given observation window T, our model for the time average landscape carbon by time T, denoted by $C(T)$ is

$$\begin{aligned} C(T) &= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T c_m(t)dt \\ &= L_a(T)+D_a(T)+D_d(T)+D_f(T)+S(T), \end{aligned} \tag{2.8}$$

where $L_a(T)$, $D_a(T)$, $D_d(T)$, $D_f(T)$, and $S(T)$ denote the average landscape carbon in the live accumulating, dead accumulating, dead decreasing, forest product, and stable pools, respectively, e.g.,

$$L_a(T) = \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T l_a(a_m(t))dt.$$

We now begin to derive the mean and variance of $L_a(T)$, $D_a(T)$, $D_d(T)$, $D_f(T)$, $S(T)$, and $C(T)$ for a landscape without disturbance as well as regular and random disturbance regimes.

2.3.2 No Disturbance Regime

Because there is no randomness in the case of no disturbance, the mean of the average landscape carbon is the average landscape carbon itself and the variance of the average landscape carbon is zero. In addition, since disturbance creates the input to the dead decreasing and forest product pools, the landscape means of these pools are zero with no disturbance. Therefore, the average landscape carbon contains only carbon in the live accumulating pool, the dead accumulating pool, and the stable pool, and is given by

$$
\begin{aligned}
C(T) &= \lim_{T\to\infty} \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M} |l_m| \int_0^T c_m(t)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M} |l_m| \int_0^T [f(a_m(t)) + g(a_m(t)) + s_m(t)]dt \\
&= \lim_{T\to\infty} \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M} |l_m| \int_0^T [f(a_m(0)+t) + g(a_m(0)+t) \\
&\quad + \frac{\eta}{\gamma} g(a_m(0)+t)]dt. \qquad (2.9)
\end{aligned}
$$

2.3.3 Regular Disturbance Regime

Similar to the case of no disturbance, the regular disturbance has no randomness either; hence, the mean of the average landscape carbon is the average landscape carbon itself and the variance of the average landscape carbon is zero. A disturbance regime is said to be regular if it has constant rotation interval R. That is

$$\tau_1^{(m)} = R - a_m(0) \text{ and } \tau_j^{(m)} = R = \text{a constant}, \quad \text{for all } 1 \leq m \leq M \text{ and } j \geq 2.$$

Assuming for simplicity that the observation window is a multiple integers of R, which is a requirement of creating a regulated age-class structure, i.e.,

$$T = kR \text{ for } k = 0, 1, 2, ...$$

Note that in this case we have,

$$T - t_{N_m(T)} = T - (kR - a_m(0)) = a_m(0).$$

Then the average landscape live accumulating carbon is derived as follows:

$$\begin{aligned}
L_a(T) &= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T f(a_m(t))dt \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\int_0^{R-a_m(0)} f(a_m(0)+t)dt + \sum_{j=2}^{N_m(T)}\int_0^R f(t)dt + \int_0^{T-t_{N_m(T)}} f(t)dt\right) \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\int_0^{R-a_m(0)} f(a_m(0)+t)dt + \sum_{j=2}^{k}\int_0^R f(t)dt + \int_0^{a_m(0)} f(t)dt\right) \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\int_{a_m(0)}^{R} f(s)ds + (k-1)\int_0^R f(t)dt + \int_0^{a_m(0)} f(t)dt\right) \\
&= \frac{1}{T}\left(\int_0^R l_a(s)ds + (k-1)\int_0^R f(t)dt\right) \\
&= \frac{1}{R}\int_0^R f(t)dt. \qquad (2.10)
\end{aligned}$$

Similarly, the average landscape dead accumulating carbon is

$$\begin{aligned}
D_a(T) &= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T g(a_m(t))dt \\
&= \frac{1}{R}\int_0^R g(t)dt. \qquad (2.11)
\end{aligned}$$

The average landscape dead decreasing carbon is

$$
\begin{aligned}
D_d(T) &= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T d_d(t)dt \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\int_0^T \sum_{j=1}^{N_m(t)} \mu[f(\tau_j^{(m)}) + g(\tau_j^{(m)})]e^{-c(t-t_j^{(m)})}\right) \\
&= \frac{1}{T}\left(\sum_{j=1}^{k}\int_0^{T-jR} \mu[f(R) + d_a(R)]e^{-ct}dt\right) \\
&= \frac{1}{T}\left(\mu[f(R) + g(R)]\sum_{j=1}^{k}(\frac{1}{c} - \frac{e^{-c(T-jR)}}{c})\right) \\
&= \frac{1}{T}\left(\mu[f(R) + g(R)]\frac{k}{c} - \mu[f(R) + g(R)]\sum_{j=1}^{k}\frac{e^{-c(T-jR)}}{c}\right), \\
&\quad \text{using sum of geometric series,} \\
&= \frac{1}{T}\left(\mu[f(R) + g(R)]\frac{T}{cR} - \mu[f(R) + g(R)]\frac{e^{cR} - e^{-c(T-R)}}{c(e^{cR} - 1)}\right) \\
&= \frac{1}{T}\left(\mu[f(R) + g(R)](\frac{T}{cR} - \frac{e^{cR} - e^{-c(T-R)}}{c(e^{cR} - 1)})\right). \qquad (2.12)
\end{aligned}
$$

Similarly, the average landscape forest product carbon is

$$
D_f(T) = \frac{1}{T}\left(\beta[f(R) + g(R)](\frac{T}{dR} - \frac{e^{dR} - e^{-d(T-R)}}{d(e^{dR} - 1)})\right) \qquad (2.13)
$$

The average landscape stable carbon is derived in terms of the the proportional combination of the dead accumulating carbon and dead decreasing carbon and hence, is given by

$$
S(T) = \frac{\eta}{\gamma}(D_a(T) + D_d(T)). \qquad (2.14)
$$

Consequently, the average landscape carbon in the regular disturbance regime, denoted by $C_{reg}(T)$, is,

$$
C_{reg}(T) = L_a(T) + D_a(T) + D_d(T) + D_f(T) + S(T). \qquad (2.15)
$$

2.3.4 Random Disturbance Regime

Not all landscapes have regular disturbance regimes. The negative exponential distribution of times between successive disturbances is commonly used to explore the recurrence of natural disturbances on a landscape (Van Wagner, 1978; Johnson and Van Wagner, 1985; Johnson and Gutsell, 1994). We will use it in our study. That is we assume the inter-arrival times of disturbances in the m^{th} stand, denoted by

$$\{\tau_j^{(m)}\}_{j=1}^{\infty},$$

are independent and identically distributed random variables having an exponential distribution with parameter λ_m.

Under this random disturbance regime, the derivation of the mean and variance of the average landscape carbon is more complicated than in the case of regulated disturbance regime.

Remark: The time average of the live and dead accumulating pools can be written approximately in terms of random sum of independent and identically distributed (i.i.d.) random variables depending on the inter-arrival times of disturbances. There are well known results from probability that allow one to quickly estimate the mean of these pools. For example, the strong law of large numbers can be used to obtain the mean of the accumulating pool in the case of infinite observation window, T. That is

$$\begin{aligned}
\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T f(a_m(t))dt\right] &= \lim_{T\to\infty}\frac{1}{T}\left[\sum_{j=1}^{N(T)}\int_0^{\tau_j} f(t)dt + \int_0^{T-t_{N(T)}} f(t)dt\right] \\
&= \lim_{T\to\infty}\left[\frac{N(T)}{T}\frac{1}{N(T)}\sum_{j=1}^{N(T)}\int_0^{\tau_j} f(t)dt\right] \\
&= \lim_{T\to\infty}\frac{N(T)}{T}\lim_{T\to\infty}\left[\frac{1}{N(T)}\sum_{j=1}^{N(T)}\int_0^{\tau_j} f(t)dt\right] \\
&= \lambda\lim_{n\to\infty}\left[\frac{1}{n}\sum_{j=1}^{n}\int_0^{\tau_j} f(t)dt\right] \\
&= \lambda E\left[\int_0^{\tau_1} f(t)dt\right], \quad \text{with probability 1.} \qquad (2.16)
\end{aligned}$$

Here τ_1 is a random variable having an exponential distribution with parameter λ. In addition, Wald's equation (Grimmett and Stirzaker, 2001) can be used to approximate the mean of this pool for a finite observation window T as follows:

$$\begin{aligned}
E\left[\frac{1}{T}\int_0^T f(a_m(t))dt\right] &= E\left[\sum_{j=1}^{N(T)}\int_0^{\tau_j} f(t)dt + \int_0^{T-t_{N(T)}} f(t)dt\right] \\
&\approx \frac{1}{T}E\left[\sum_{j=1}^{N(T)+1}\int_0^{\tau_j} f(t)dt\right] \\
&= \frac{1}{T}E[N(T)+1]E\left[\int_0^{\tau_1} f(t)dt\right] \\
&= \frac{1}{T}(\lambda T+1)E\left[\int_0^{\tau_1} f(t)dt\right]. \qquad (2.17)
\end{aligned}$$

Obviously, as $T \to \infty$ this result is equivalent to the result obtained by the strong law of large numbers.

The result from exchangeable random variables can also be used to approximate the mean and variance of the live and dead accumulating pools but they require the concept from order statistics of random variables (Ross, 2002).

The dead decreasing and forest product pools are not only function of random sum of random i.i.d. inter-arrival times but also random arrival times of disturbances. There are no obvious results that we know can give a quick estimate for the mean and variance of these pools.

Before developing a general theorem from which the mean and variance of all the pools of landscape carbon can be obtained, it is important to recall that each of the pool as well as the total pool of the landscape carbon can be written in the following form:

$$\frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)}) + H(T, t_{N_m(T)})\right),$$

with appropriate functions G and H. For example, the live accumulating landscape carbon can be written as

$$\begin{aligned}
& \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T f(a_m(t))dt \\
= & \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\sum_{j=1}^{N_m(T)}\int_0^{\tau_j^{(m)}} f(t)dt + \int_0^{T-t_{N_m(T)}} f(t)dt \\
= & \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)}) + G(T, t_{N_m(T)})\right),
\end{aligned}$$

where

$$G(s,u) = \int_0^{u-s} f(t)dt.$$

Similarly, the dead decreasing pool can written be as

$$\begin{aligned}
& \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T d(t)dt \\
= & \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\sum_{j=1}^{N_m(t)}\int_0^{T-t_j^{(m)}} \mu[f(\tau_j^{(m)}) + g(\tau_j^{(m)})]e^{-c(t)}dt \\
= & \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\sum_{j=1}^{N_m(t)}\int_0^{T-t_j^{(m)}} \mu[f(t_j^{(m)} - t_{j-1}^{(m)}) + g(t_j^{(m)} - t_{j-1}^{(m)})]e^{-c(t)}dt \\
= & \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)})\right),
\end{aligned}$$

where

$$G(s,u) = \int_0^{T-u} \mu[f(u-s) + g(u-s)]e^{-c(t)}dt.$$

In addition, one of the properties of expectation is that the expectation of the sum is the sum of expectation and hence,

$$
\begin{aligned}
& E\left[\frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)}) + H(T, t_{N_m(T)})\right)\right] \\
= & \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m| E\left[\left(\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)}) + H(T, t_{N_m(T)})\right)\right]. \qquad (2.18)
\end{aligned}
$$

Also, by assuming that stands are independent of each other, from the property of variance of sum of independent variables, one has

$$
\begin{aligned}
& \mathrm{Var}\left[\frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\left(\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)}) + H(T, t_{N_m(T)})\right)\right] \\
= & \frac{1}{T^2}\frac{1}{|L^2|}\sum_{m=1}^{M}|l_m|^2 \mathrm{Var}\left[\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)}) + H(T, t_{N_m(T)})\right] \qquad (2.19) \\
= & \frac{1}{T^2}\frac{1}{|L^2|}\sum_{m=1}^{M}|l_m|^2 \left(E\left[\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)}) + H(T, t_{N_m(T)})\right]^2\right. \\
& \left. - \left(E\left[\sum_{j=1}^{N_m(T)} G(t_{j-1}^{(m)}, t_j^{(m)}) + H(T, t_{N_m(T)})\right]\right)^2\right). \qquad (2.20)
\end{aligned}
$$

Therefore, the quantities of importance in obtaining the mean and variance of all the landscape carbon pools, suppressing the index m for ease of notation, are

$$
E\left[\sum_{j=1}^{N(T)} G(t_{j-1}, t_j) + H(T, t_{N(T)})\right],
$$

and

$$
E\left[\sum_{j=1}^{N(T)} G(t_{j-1}, t_j) + H(T, t_{N(T)})\right]^2.
$$

We now introduce the following well known results in probability from which will be used in obtaining these two quantities. The index m is suppressed for the following derivation.

Tower property of expectation (Ross, 2002): Let X and Y be any random variables then

$$
E[X] = E[E[X|Y]].
$$

Relation between exponential and Poisson random variables (Ross, 2002): Let

$$\{\tau_j\}_{j=1}^{\infty},$$

be a sequence of independent and identically distributed random variables having an exponential distribution with parameter λ.

Let

$$N(t) = min\{k : \tau_1 + \tau_2 + ... + \tau_k \leq t < \tau_1 + \tau_2 + ... + \tau_k + \tau_{k+1}\}.$$

Then $N(T)$ is a Poisson random variable with parameter λt. That is

$$P\{N(t) = k\} = e^{-\lambda T} \frac{(\lambda t)^k}{k!}.$$

Relation to order statistics (Grimmett and Stirzaker, 2001): With the above notations and let

$$t_j = \sum_{i=1}^{j} \tau_i, \quad \text{for all } j \geq 1, \quad \text{with } t_0 = 0.$$

Then given that $\{N(t) = k\}$, the k arrival times $t_1, ..., t_k$ have the same distribution as the order statistics corresponding to k independent random variables uniformly distributed on the interval $(0, t)$.

In the application $\{\tau_j\}_{j=1}^{\infty}$ denote the times between successive disturbances, $\{t_j\}_{j=1}^{\infty}$ denote the arrival times of disturbances, and $N(T)$ denotes the number of disturbances that have occurred by time T.

We now briefly introduce the notations of the order statistics of random variables, details of this section can be found in (Ross, 2002). Let $X_1, X_2, ..., X_k$ be k independent and identically distributed, continuous random variables having a common density f and distribution function F. Define

$$\begin{aligned}
X_{(1)} &= \text{smallest of } X_1, X_2, ..., X_k \\
X_{(2)} &= \text{second smallest of } X_1, X_2, ..., X_k \\
&. \\
&. \\
&. \\
X_{(j)} &= j^t h \text{ smallest of } X_1, X_2, ..., X_k \\
&. \\
&. \\
&. \\
X_{(k)} &= \text{largest of } X_1, X_2, ..., X_k
\end{aligned}$$

The ordered values $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(k)}$ are known as the order statistics corresponding to the random variables $X_1, X_2, ..., X_k$. The density function of the j^{th} order statistic $X_{(j)}$ can be obtained by reasoning as follows: In order for $X_{(j)}$ equal to x, it is necessary for $j-1$ of the k values X_1,...,X_k to be less than x, $k-j$ of them be greater than x, and 1 of them equal to x. Now, the probability density that any given set of $j-1$ of the $X_i's$ are less than x, another given set of $k-j$ are all greater than x, and the remaining value is equal to x, equals

$$[F(x)]^{j-1}[1-F(x)]^{k-j}f(x)$$

Hence, as there are

$$\frac{k!}{(k-j)!(j-1)!}$$

different partitions of the k random variables X_1,...,X_k into the three groups, we see that the density function of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{k!}{(j-1)!(k-j)!}[F(x)]^{j-1}[1-F(x)]^{k-j}f(x). \tag{2.21}$$

For $1 \leq i < j \leq k$ and $1 \leq i < j < m \leq k$ the following joint density functions can be derived in similar reasoning,

$$\begin{aligned} f_{X_{(i)},X_{(j)}}(x_i, x_j) &= \frac{k!}{(i-1)!(j-i-1)!(k-j)!}[[F(x_i)]^{i-1} \\ &\quad [F(x_j)-F(x_i)]^{j-i-1}[1-F(x_j)]^{k-j}f(x_i)f(x_j)], \qquad (2.22) \\ &\quad \text{for all } x_i < x_j, \end{aligned}$$

and

$$\begin{aligned} f_{X_{(i)},X_{(j)},X_{(m)}}(x_i, x_j, x_m) &= \frac{k!}{(i-1)!(j-i-1)!(m-j-1)!(k-m)!}[[F(x_i)]^{i-1} \\ &\quad [F(x_j)-F(x_i)]^{j-i-1}[F(x_m)-F(x_j)]^{m-j-1}[1-F(x_m)]^{k-m} \\ &\quad f(x_i)f(x_j)f(x_m)], \qquad (2.23) \\ &\quad \text{for all } x_i < x_j < x_m. \end{aligned}$$

In fact, any joint density function of the order statistics can be derived by employing similar reasoning.

Lemma 1 *Let t_1,...,t_k denote the order statistics of k random variables X_1,...,X_k which are independent and uniformly distributed on the interval $(0, t)$ then the following joint density functions are given as follows:*

$$f_{t_j}(x) = \frac{k!}{(j-1)!(k-j)!t^k}[x]^{j-1}[t-x]^{k-j}. \tag{2.24}$$

$$f_{t_i,t_j}(x, y) = \frac{k!}{(i-1)!(j-i-1)!(k-j)!t^k}[x]^{i-1}[y-x]^{j-i-1}[t-y]^{k-j} \tag{2.25}$$

Proof:

$$\begin{aligned}
f_{t_j}(x) &= \frac{k!}{(j-1)!(k-j)!}[F(x)]^{j-1}[1-F(x)]^{k-j}f(x) \\
&= \frac{k!}{(j-1)!(k-j)!}[\frac{x}{t}]^{j-1}[1-\frac{x}{t}]^{k-j}\frac{1}{t} \\
&= \frac{k!}{(j-1)!(k-j)!t^k}[x]^{j-1}[t-x]^{k-j}.
\end{aligned}$$

$$\begin{aligned}
f_{t_i,t_j}(x,y) &= \frac{k!}{(i-1)!(j-i-1)!(k-j)!}[[F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-i-1} \\
&\quad [1-F(x_j)]^{k-j}f(x_i)f(x_j)] \\
&= \frac{k!}{(i-1)!(j-i-1)!(k-j)!}[\frac{x}{t}]^{i-1}[\frac{y}{t}-\frac{x}{t}]^{j-i-1}[1-\frac{y}{t}]^{k-j}\frac{1}{t}\frac{1}{t} \\
&= \frac{k!}{(i-1)!(j-i-1)!(k-j)!t^k}[x]^{i-1}[y-x]^{j-i-1}[t-y]^{k-j}. \quad \blacksquare
\end{aligned}$$

Before going to the next lemma we introduce a function known as the indicator function. The indicator function of a subset A of a set X is a function

$$\mathbf{1}_A : X \to \{0,1\}$$

as defined

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Lemma 2 *Let H be a continuous function, then*

$$E\left[H(t, t_{N(t)})\right] = e^{-\lambda t}H(t,0) + \int_0^t \lambda e^{-\lambda(t-x)}H(t,x)dx. \tag{2.26}$$

Proof: Conditioning on the event $\{N(t)=k\}$ we have,

$$\begin{aligned}
E[H(t,t_{N(t)})] &= E[H(t,t_{N(t)})\mathbf{1}_{[N(t)=0]}] + E[H(t,t_{N(t)})\mathbf{1}_{[N(t)\geq 1]}] \\
&= e^{-\lambda t}H(t,0) + E\left[E\left[H(t,t_{N(t)})\mathbf{1}_{[N(t)\geq 1]}|N(t)\right]\right] \\
&= e^{-\lambda t}H(t,0) + \sum_{k=1}^{\infty} P(N(t)=k)E\left[H(t,t_{N(t)})|N(t)=k\right] \\
&= e^{-\lambda t}H(t,0) + \sum_{k=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\int_0^t f_{t_k}(x)H(t,x)dx \\
&= e^{-\lambda t}H(t,0) + \sum_{k=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\int_0^t \frac{kx^{k-1}}{t^k}H(t,x)dx \\
&= e^{-\lambda t}H(t,0) + \int_0^t \sum_{k=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\frac{kx^{k-1}}{t^k}H(t,x)dx \\
&= e^{-\lambda t}H(t,0) + \int_0^t \lambda e^{-\lambda(t-x)}H(t,x)dx. \quad \blacksquare
\end{aligned}$$

Proposition 1 *Let G be a continuous function, then*

$$E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)\mathbf{1}_{[N(t)\geq 1]}\right] = \int_0^t \lambda e^{-\lambda x} G(0,x)dx + \int_0^t \int_0^y G(x,y)\lambda^2 e^{-\lambda(y-x)}dxdy. \tag{2.27}$$

Furthermore,

$$\begin{aligned} & E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j)\right)^2 \mathbf{1}_{[N(t)\geq 1]}\right] \\ = & \int_0^t \lambda e^{-\lambda x} G^2(0,x)dx + \int_0^t \int_0^y G^2(x,y)\lambda^2 e^{-\lambda(y-x)}dxdy \\ & +2\int_0^t \int_0^y \lambda^2 e^{-\lambda y} G(0,x)G(x,y)dxdy \\ & +2\int_0^t \int_0^z \int_0^y \lambda^3 e^{-\lambda(x-y+z)} G(0,x)G(y,z)dxdydz \\ & +2\int_0^t \int_0^z \int_0^y \lambda^3 e^{-\lambda(z-x)} G(x,y)G(y,z)dxdydz \\ & +2\int_0^t \int_0^w \int_0^z \int_0^y \lambda^4 e^{-\lambda(w-x+y-z)} G(x,y)G(z,w)dxdydzdw. \end{aligned}$$

Proof: Conditioning on then event $\{N(t)=k\}$, we have

$$
\begin{aligned}
& E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)\mathbf{1}_{[N(t)\geq 1]}\right] \\
= \;& E\left[E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)\mathbf{1}_{[N(t)\geq 1]}|N(t)\right]\right] \\
= \;& \sum_{k=1}^{\infty} P(\{N(t)=k\})E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)|N(t)=k\right] \\
= \;& \sum_{k=1}^{\infty} P(\{N(t)=k\})\sum_{j=1}^{k} E[G(t_{j-1},t_j)|N(t)=k] \\
= \;& \sum_{k=1}^{\infty} e^{(-\lambda t)}\frac{(\lambda t)^k}{k!}\left[\int_0^t f_{t_1}(x)G(0,x)dx + \sum_{j=2}^{k}\int_0^t\int_0^u f_{t_{j-1},t_j}(s,u)G(s,u)dsdu\right] \\
= \;& \sum_{k=1}^{\infty} e^{(-\lambda t)}\frac{(\lambda t)^k}{k!}\left[\int_0^t \frac{k(t-x)^{k-1}}{t^k}G(0,x)dx + \sum_{j=2}^{k}\int_0^t\int_0^u \frac{k!(s)^{j-2}(t-u)^{k-j}}{(j-2)!(k-j)!t^k}G(s,u)dsdu\right] \\
= \;& e^{-\lambda t}\int_0^t \lambda e^{\lambda(t-x)}G(0,x)dx + e^{-\lambda t}\sum_{k=2}^{\infty}\lambda^k\sum_{j=2}^{k}\int_0^t\int_0^u \frac{(s)^{j-2}(t-u)^{k-j}G(s,u)}{(j-2)!(k-j)!}dsdu \\
= \;& e^{-\lambda t}\int_0^t \lambda e^{\lambda(t-x)}G(0,x)dx + e^{-\lambda t}\int_0^t\int_0^u \sum_{k=2}^{\infty}\lambda^k\sum_{j=2}^{k}\frac{(s)^{j-2}(t-u)^{k-j}G(s,u)}{(j-2)!(k-j)!}dsdu \\
& \text{interchanging the order of summations} \\
= \;& e^{-\lambda t}\int_0^t \lambda e^{\lambda(t-x)}G(0,x)dx + e^{-\lambda t}\int_0^t\int_0^u G(s,u)\sum_{j=2}^{\infty}\frac{s^{j-2}}{(j-2)!}\sum_{k=j}^{\infty}\frac{\lambda^k(t-u)^{k-j}}{(k-j)!}dsdu \\
= \;& e^{-\lambda t}\int_0^t \lambda e^{\lambda(t-x)}G(0,x)dx + e^{-\lambda t}\int_0^t\int_0^u G(s,u)\sum_{j=2}^{\infty}\lambda^2\frac{\lambda^{j-2}s^{j-2}}{(j-2)!}\sum_{k=j}^{\infty}\frac{\lambda^{k-j}(t-u)^{k-j}}{(k-j)!}dsdu \\
= \;& e^{-\lambda t}\int_0^t \lambda e^{\lambda(t-x)}G(0,x)dx + e^{-\lambda t}\int_0^t\int_0^u G(s,u)\lambda^2 e^{\lambda s}e^{\lambda(t-u)}dsdu \\
= \;& \int_0^t \lambda e^{-\lambda x}G(0,x)dx + \int_0^t\int_0^u G(s,u)\lambda^2 e^{-\lambda(u-s)}dsdu.
\end{aligned}
$$

Again, conditioning on the event $\{N(t) = k\}$, we have

$$
\begin{aligned}
& E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1}, t_j)\right)^2 \mathbf{1}_{[N(t)\geq 1]}\right] \\
= \;& E\left[E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1}, t_j)\right)^2 \mathbf{1}_{[N(t)\geq 1]} | N(t)\right]\right] \\
= \;& E\left[E\left[\left(\sum_{j=1}^{N(t)} G^2(t_{j-1}, t_j) + \sum_{j\neq\ i=1}^{N(t)} G(t_{i-1}, t_i)G(t_{j-1}, t_j)\right) \mathbf{1}_{[N(t)\geq 1]} | N(t)\right]\right] \\
= \;& E\left[E\left[\left(\sum_{j=1}^{N(t)} G^2(t_{j-1}, t_j) + 2\sum_{j=1}^{N(t)}\sum_{i=1}^{j-1} G(t_{i-1}, t_i)G(t_{j-1}, t_j)\right) \mathbf{1}_{[N(t)\geq 1]} | N(t)\right]\right] \\
= \;& E\left[E\left[\sum_{j=1}^{N(t)} G^2(t_{j-1}, t_j)\mathbf{1}_{[N(t)\geq 1]} + 2\sum_{j=4}^{N(t)}\sum_{i=2}^{j-2} G(t_{i-1}, t_i)G(t_{j-1}, t_j)\mathbf{1}_{[N(t)\geq 4]}\right.\right. \\
& \left.\left. + \; 2\sum_{j=2}^{N(t)} G(0, t_1)G(t_{j-1}, t_j)\mathbf{1}_{[N(t)\geq 2]} + 2\sum_{i=2}^{N(t)-1} G(t_{i-1}, t_i)G(t_i, t_{i+1})\mathbf{1}_{[N(t)\geq 3]}) | N(t)\right]\right]
\end{aligned}
$$

Now, by the preceding part we have

$$
\begin{aligned}
& E\left[E\left[\sum_{j=1}^{N(t)} G^2(t_{j-1}, t_j)\mathbf{1}_{[N(t)\geq 1]} | N(t)\right]\right] \\
= \;& \int_0^t \lambda e^{-\lambda x} G^2(0, x)dx + \int_0^t \int_0^u G^2(s, u)\lambda^2 e^{-\lambda(u-s)} ds du.
\end{aligned}
$$

$$
\begin{aligned}
& E\left[E\left[\sum_{j=2}^{N(t)} G(t_0,t_1)G(t_{j-1},t_j)\mathbf{1}_{[N(t)\geq 2]}|N(t)\right]\right] \\
= \;& E\left[E\left[\left(G(t_0,t_1)G(t_1,t_2)+\sum_{j=3}^{N(t)} G(t_0,t_1)G(t_{j-1},t_j)\right)\mathbf{1}_{[N(t)\geq 2]}|N(t)\right]\right] \\
= \;& \sum_{k=2}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left[E\left[G(t_0,t_1)G(t_1,t_2)+\sum_{j=3}^{N(t)} G(t_0,t_1)G(t_{j-1},t_j)|N(t)=k\right]\right] \\
= \;& \sum_{k=2}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left[\int_0^t\int_0^y f_{t_1,t_2}(x,y)(G(0,x)G(x,y)dxdy\right. \\
& \left. +\sum_{j=3}^{k}\int_0^t\int_0^z\int_0^y f_{t_1,t_{j-1},t_j}(x,y,z)G(0,x)G(y,z)dxdydz\right] \\
= \;& \sum_{k=2}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left[\int_0^t\int_0^y \frac{k!}{(k-2)!t^k}[t-y]^{k-2}(G(0,x)G(x,y)dxdy\right. \\
& \left. +\sum_{j=3}^{k}\int_0^t\int_0^z\int_0^y \frac{k![y-x]^{(j-3)}[t-z]^{k-j}}{(j-3)!(k-j)!}\frac{1}{t^k}G(0,x)G(y,z)dxdydz\right],
\end{aligned}
$$

interchanging the order of summations as the preceding part,

$$
\begin{aligned}
= \;& e^{-\lambda t}\left[\int_0^t\int_0^y \lambda^2 e^{\lambda(t-y)}G(0,x)G(x,y)dxdy\right. \\
+ \;& \left.\int_0^t\int_0^z\int_0^y \lambda^3 e^{\lambda(y-x+t-z)}G(t_0,x)G(y,z)dxdydz\right] \\
= \;& \int_0^t\int_0^y \lambda^2 e^{-\lambda y}G(0,x)G(x,y)dxdy \\
+ \;& \int_0^t\int_0^z\int_0^y \lambda^3 e^{-\lambda(x-y+z)}G(0,x)G(y,z)dxdydz.
\end{aligned}
$$

$$
\begin{aligned}
& E\left[E\left[\sum_{j=2}^{N(t)-1} G(t_{i-1},t_i)G(t_i,t_{i+1})\mathbf{1}_{[N(t)\geq 3]}|N(t)\right]\right] \\
= & \sum_{k=3}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sum_{j=2}^{k-1}\int_0^t\int_0^z\int_0^y f_{t_{i-1},t_i,t_{i+1}}(x,y,z)G(x,y)G(y,z)dxdydz \\
= & \sum_{k=3}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sum_{i=2}^{k-1}\int_0^t\int_0^z\int_0^y \frac{k!x^{i-2}(t-z)^{k-i-1}}{t^k(i-2)!(k-i-1)!}G(x,y)G(y,z)dxdydz \\
= & e^{-\lambda t}\int_0^t\int_0^z\int_0^y \lambda^3 e^{\lambda(x+t-z)}G(x,y)G(y,z)dxdydz \\
= & \int_0^z\int_0^y \lambda^3 e^{-\lambda(z-x)}G(x,y)G(y,z)dxdydz.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& E\left[E\left[\sum_{j=4}^{k}\sum_{i=2}^{j-2} G(t_{i-1},t_i)G(t_{j-1},t_j)\mathbf{1}_{[N(t)\geq 4]}|N(t)\right]\right] \\
= & \sum_{k=4}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sum_{j=4}^{k}\sum_{i=2}^{j-2}\int_0^t\int_0^w\int_0^z\int_0^y f_{t_{i-1},t_i,t_{j-1},t_j}(x,y,z,w)G(x,y)G(z,w))dxdydzdw \\
= & \sum_{k=4}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sum_{j=4}^{k}\sum_{i=2}^{j-2}\int_0^t\int_0^w\int_0^z\int_0^y \frac{k!(x)^{i-2}(z-y)^{j-i-2}(t-w)^{k-j}}{t^k(k-j)!(i-2)!(j-i-2)!} \\
& G(x,y)G(z,w)dxdydzdw \\
= & e^{-\lambda t}\int_0^t\int_0^w\int_0^z\int_0^y \lambda^4 e^{\lambda(t-w+x-y+z)}G(x,y)G(z,w)dxdydzdw \\
= & \int_0^t\int_0^w\int_0^z\int_0^y \lambda^4 e^{-\lambda(y-x+w-z)}G(x,y)G(z,w)dxdydzdw. \quad \blacksquare
\end{aligned}
$$

Lemma 3 *Let G and H be continuous functions, then*

$$\begin{aligned}
& E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)H(t,t_{N(t)})\boldsymbol{1}_{[N(t)\geq 1]}\right] \\
= \; & \lambda e^{-\lambda t}\int_0^t G(0,x)H(t,x)dx + \int_0^t\int_0^y \lambda^2 e^{-\lambda(t+x-y)}G(0,x)H(t,y)dxdy \\
& + \int_0^t\int_0^z\int_0^y \lambda^3 e^{-\lambda(t+x-y+z)}H(x,y)G(t,z)dxdydz \\
& + \int_0^t\int_0^y \lambda e^{-\lambda(t-x)}G(x,y)H(t,y)dxdy. \qquad (2.28)
\end{aligned}$$

Proof: Conditioning on the event $\{N(t)=k\}$, we have

$$
\begin{aligned}
& E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)H(t,t_{N(t)})\mathbf{1}_{[N(t)\geq 1]}\right] \\
= \;& E\left[E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)H(t,t_{N(t)})\mathbf{1}_{[N(t)\geq 1]}|N(t)\right]\right] \\
= \;& \sum_{k=1}^{\infty} P(N(t)=k)E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)H(t,t_{N(t)})|N(t)=k\right] \\
= \;& E[G(0,t_1)G(t,t_{N(t)})|N(t)=1] \\
& +\sum_{k=2}^{\infty} P(N(t)=k)E\left[\left(G(0,t_1)H(t,t_{N(t)})+\sum_{j=2}^{N(t)-1} G(t_{j-1},t_j)H(t,t_{N(t)})\right.\right. \\
& \left.\left.+G(t_{N(t)-1},t_{N(t)})H(t,t_{N(t)})\right)|N(t)=k\right] \\
= \;& \lambda t e^{-\lambda t}\frac{1}{t}\int_0^t G(0,x)H(t,x)dx+\sum_{k=2}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left[\left(\int_0^t\int_0^y f_{t_1,t_k}(x,y)G(0,x)H(t,y)dxdy\right.\right. \\
& +\sum_{j=2}^{k-1}\int_0^t\int_0^z\int_0^y f_{t_{j-1},t_j,t_k}(x,y,z)G(x,y)H(t,z)dxdydz \\
& \left.\left.+\int_0^t\int_0^y f_{t_{k-1},t_k}(x,y)G(x,y)H(t,y)dxdy\right)\right] \\
= \;& \lambda e^{-\lambda t}\int_0^t G(0,x)H(t,x)dx \\
& +\sum_{k=2}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left[\left(\int_0^t\int_0^y \frac{k!}{(k-2)!t^k}(y-x)^{k-2}G(0,x)H(t,y)dxdy\right.\right. \\
& +\sum_{j=2}^{k-1}\int_0^t\int_0^z\int_0^y \frac{k!}{(j-2)!(k-j-1)!t^k}x^{j-2}(z-y)^{k-j-1}G(x,y)H(t,z)dxdydz \\
& \left.\left.+\int_0^t\int_0^y \frac{k!}{(k-2)!t^k}x^{k-2}G(x,y)H(t,y)dxdy\right)\right] \\
= \;& \lambda e^{-\lambda t}\int_0^t G(0,x)H(t,x)dx+\int_0^t\int_0^y \lambda^2 e^{-\lambda(t+x-y)}G(0,x)H(t,y)dxdy \\
& +\int_0^t\int_0^z\int_0^y \lambda^3 e^{-\lambda(t+x-y+z)}G(x,y)H(t,z)dxdydz \\
& +\int_0^t\int_0^y \lambda e^{-\lambda(t-x)}G(x,y)H(t,y)dxdy. \;\blacksquare
\end{aligned}
$$

Theorem 1 *Let G and H be continuous functions then*

$$\begin{aligned} & E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right] \\ = \quad & \int_0^t \lambda e^{-\lambda x} G(0,x)dx + \int_0^t \int_0^y G(x,y)\lambda^2 e^{-\lambda(y-x)}dxdy \\ & + \int_0^t \lambda e^{-\lambda(t-x)} H(t,x)dx + e^{-\lambda t}H(t,0). \end{aligned} \tag{2.29}$$

Furthermore,

$$\begin{aligned} & E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right)^2 \mathbf{1}_{[N(t)\geq 1]}\right] \\ = \quad & \int_0^t \lambda e^{-\lambda x} G^2(0,x)dx + \int_0^t \int_0^y G^2(x,y)\lambda^2 e^{-\lambda(y-x)}dxdy \\ & +2\int_0^t \int_0^y \lambda^2 e^{-\lambda y} G(0,x)G(x,y)dxdy + 2\int_0^t\int_0^z\int_0^y \lambda^3 e^{-\lambda(x-y+z)}G(0,x)G(y,z)dxdydz \\ & +2\int_0^t\int_0^z\int_0^y \lambda^3 e^{-\lambda(z-x)}G(x,y)G(y,z)dxdydz \\ & +2\int_0^t\int_0^w\int_0^z\int_0^y \lambda^4 e^{-\lambda(w-x+y-z)}G(x,y)G(z,w)dxdydzdw. \\ & +2\lambda e^{-\lambda t}\int_0^t G(0,x)H(t,x)dx + 2\int_0^t\int_0^y \lambda^2 e^{-\lambda(t+x-y)}G(0,x)H(t,y)dxdy \\ & +2\int_0^t\int_0^z\int_0^y \lambda^3 e^{-\lambda(t+x-y+z)}G(x,y)H(t,z)dxdydz \\ & +2\int_0^t\int_0^y \lambda e^{-\lambda(t-x)}G(x,y)H(t,y)dxdy + \int_0^t \lambda e^{-\lambda(t-x)}H^2(t,x)dx. \end{aligned}$$

Consequently,

$$\begin{aligned} & \mathrm{Var}\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right) \mathbf{1}_{[N(t)\geq 1]}\right] \\ = \quad & E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right)^2 \mathbf{1}_{[N(t)\geq 1]}\right] \\ & -\left(E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right) \mathbf{1}_{[N(t)\geq 1]}\right]\right)^2. \end{aligned} \tag{2.30}$$

Proof:

$$
\begin{aligned}
& E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right] \\
= \;& E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right)\mathbf{1}_{[N(t)\geq 1]}\right] + E\left[H(t,t_{N(t)})\mathbf{1}_{[N(t)=0]}\right] \\
= \;& E\left[\sum_{j=1}^{N(t)} G(t_{j-1},t_j)\mathbf{1}_{[N(t)\geq 1]}\right] + E\left[\sum_{j=1}^{N(t)} H(t,t_{N(t)})\mathbf{1}_{[N(t)\geq 1]}\right] + E\left[H(t,t_{N(t)})\mathbf{1}_{[N(t)=0]}\right] \\
= \;& \int_0^t \lambda e^{-\lambda x} G(0,x)dx + \int_0^t \int_0^y G(x,y)\lambda^2 e^{-\lambda(y-x)}dxdy \\
& + \int_0^t \lambda e^{-\lambda(t-x)} H(t,x)dx + e^{-\lambda t} H(t,0),
\end{aligned}
$$

by lemma 2 and proposition 1.

$$
\begin{aligned}
& E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right)^2 \mathbf{1}_{[N(t)\geq 1]}\right] \\
= \;& E\left[\left(\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j)\right)^2 + 2H(t,t_{N(t)})\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H^2(t,t_{N(t)})\right)\mathbf{1}_{[N(t)\geq 1]}\right],
\end{aligned}
$$

and hence, the result follows directly from lemma 2, lemma 3, and proposition 1.

Finally, by the definition of variance,

$$
\begin{aligned}
& \mathrm{Var}\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right)\mathbf{1}_{[N(t)\geq 1]}\right] \\
= \;& E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right)^2 \mathbf{1}_{[N(t)\geq 1]}\right] \\
& - \left(E\left[\left(\sum_{j=1}^{N(t)} G(t_{j-1},t_j) + H(t,t_{N(t)})\right)\mathbf{1}_{[N(t)\geq 1]}\right]\right)^2 . \quad \blacksquare
\end{aligned}
$$

We now can obtain the mean of all the pools of the average landscape carbon by applying the first part of theorem 1.

The mean landscape live accumulating carbon,

$$\begin{aligned}
E[L_a(T)] &= E\left[\frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T f(a_m(t))dt\right] \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T E\left[f(a_m(t))\right]dt, \\
&\quad \text{using the first part of theorem 1 with } G=0 \text{ and } H(t,t_{N(t)}) = f(a_m(t)), \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T \left[e^{-\lambda_m t}f(a_m(0)+t) + \int_0^t \lambda_m e^{-\lambda_m s} f(s)ds\right]dt. \qquad (2.31)
\end{aligned}$$

Similarly, the mean landscape dead accumulating carbon,

$$E[D_a(T)] = \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T \left[e^{-\lambda_m t}g(a_m(0)+t) + \int_0^t \lambda_m e^{-\lambda_m s} g(s)ds\right]dt \qquad (2.32)$$

The mean landscape dead decreasing pool,

$$\begin{aligned}
E[D_d(T)] &= E\left[\frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T d(t)\right] \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m| E\left[\int_0^T \sum_{j=1}^{N_m(t)} \mu[f(\tau_j^{(m)}) + g(\tau_j^{(m)})]e^{-c(t-t_j^{(m)})}dt\right] \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T E\left[\sum_{j=1}^{N_m(t)} \mu[f(t_j^{(m)}-t_{j-1}^{(m)}) + g(t_j^{(m)}-t_{j-1}^{(m)})]e^{-c(t-t_j^{(m)})}\right]dt \\
&= \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T \left[\int_0^t \lambda_m e^{-\lambda_m y} G_d(0,y;t)dy\right. \\
&\quad \left. + \int_0^t\int_0^y G_d(x,y;t)\lambda_m^2 e^{-\lambda_m(y-x)}dxdy\right]dt. \qquad (2.33)
\end{aligned}$$

Here, treating t as a parameter

$$G_d(x,y;t) = \mu[f(y-x) + g(y-x)]e^{-c(t-y)}.$$

The first part of theorem 1 is used with

$$G(x,y) = G_d(x,y;t), \quad \text{and} \quad H = 0.$$

Similarly, the mean landscape forest product pool

$$E[D_f(T)] = \frac{1}{T}\frac{1}{|L|}\sum_{m=1}^{M}|l_m|\int_0^T\left[\int_0^t \lambda_m e^{-\lambda_m x}G_f(0,x;t)dx + \int_0^t\int_0^y G_f(x,y;t)\lambda_m^2 e^{-\lambda_m(y-x)}dxdy\right]dt, \quad (2.34)$$

where

$$G_f(x,y;t) = \beta[f(y-x)+g(y-x)]e^{-d(t-y)}.$$

The mean landscape stable pool,

$$E[S(T)] = \frac{\eta}{\gamma}(E[D_a(T)] + E[D_d(T)]). \quad (2.35)$$

Consequently, the mean landscape carbon in the random disturbance regime, denoted by $\overline{C(T)}$, is

$$\overline{C(T)} = E[L_a(T)] + E[D_a(T)] + E[D_d(T)] + E[D_f(T)] + E[S(T)]. \quad (2.36)$$

For derivation of the variance of each of the five pools and the total pool we assume that c_1, c_2, ..., c_M are independent. We also assume that

$$a_m(0) = 0 \text{ for all } 1 \le m \le M.$$

By independence we have,

$$\begin{aligned}\mathrm{Var}[C(T)] &= \mathrm{Var}\left[\frac{1}{T}\frac{1}{L}\sum_{m=1}^{M}|l_m|\int_0^T c_m(t)dt\right] \\ &= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\int_0^T c_m(t)dt\right]. \quad (2.37)\end{aligned}$$

We first derive the variance of each pool of carbon and then derive the variance of the average landscape carbon.

The variance landscape live accumulating carbon,

$$
\begin{aligned}
\mathrm{Var}[L_a(T)] &= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\int_0^T f(a_m(t))dt\right] \\
&= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)}\int_0^{\tau_j^{(m)}} f(t)dt + \int_0^{T-T_{N_m(T)}} f(t)dt\right] \\
&= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)}\int_0^{t_j^{(m)}-t_{j-1}^{(m)}} f(t)dt + \int_0^{T-t_{N_m(T)}} f(t)dt\right] \\
&= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)} P(t_{j-1}^{(m)}, t_j^{(m)}) + Q(T, t_{N_m(T)})\right] \qquad (2.38)
\end{aligned}
$$

where

$$P(t_{j-1}^{(m)}, t_j^{(m)}) = \int_0^{t_j^{(m)}-t_{j-1}^{(m)}} f(t)dt,$$

and

$$Q(T, t_{N_m(T)}) = \int_0^{T-t_{N_m(T)}} f(t)dt.$$

Using of theorem 1 to obtain

$$Var\left[\sum_{j=1}^{N_m(T)} P(t_{j-1}^{(m)}, t_j^{(m)}) + Q(T, t_{N_m(T)})\right],$$

with

$$P = G \text{ and } Q = H.$$

Similarly, the variance landscape dead accumulating carbon,

$$\mathrm{Var}[D_a(T)] = \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)} X(t_{j-1}^{(m)}, t_j^{(m)}) + Y(T, t_{N_m(T)})\right] \qquad (2.39)$$

where

$$X(t_{j-1}^{(m)}, t_j^{(m)}) = \int_0^{t_j^{(m)}-t_{j-1}^{(m)}} g(t)dt,$$

and

$$Y(T, t_{N_m(T)}) = \int_0^{T-t_{N_m(T)}} g(t)dt.$$

The variance landscape dead decreasing carbon,

$$\begin{aligned}
\mathrm{Var}[D_d(T)] &= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\int_0^T d(t)dt\right] \\
&= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)}\int_0^{T-t_j^{(m)}}\mu[f(\tau_j^{(m)})+g(\tau_j^{(m)})]e^{-ct}dt\right] \\
&= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)}\int_0^{T-t_j^{(m)}}\mu[f(t_j^{(m)}-t_{j-1}^{(m)})+g(t_j^{(m)}-t_{j-1}^{(m)})]e^{-ct}dt\right] \\
&= \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)}W(t_{j-1}^{(m)},t_j^{(m)})\right], \qquad (2.40)
\end{aligned}$$

where

$$W(t_{j-1}^{(m)},t_j^{(m)}) = \int_0^{T-t_j^{(m)}}\mu[f(t_j^{(m)}-t_{j-1}^{(m)})+g(t_j^{(m)}-t_{j-1}^{(m)})]e^{-ct}dt.$$

Using theorem 1 to obtain

$$\mathrm{Var}\left[\sum_{j=1}^{N_m(T)}W(t_{j-1}^{(m)},t_j^{(m)})\right],$$

with

$$W = G \text{ and } H = 0.$$

Similarly, the variance landscape forest product carbon,

$$\mathrm{Var}[D_f(T)] = \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)}J(t_{j-1}^{(m)},t_j^{(m)})\right], \qquad (2.41)$$

where

$$J(t_{j-1}^{(m)},t_j^{(m)}) = \int_0^{T-t_j^{(m)}}\beta[f(t_j^{(m)}-t_{j-1}^{(m)})+g(t_j^{(m)}-t_{j-1}^{(m)})]e^{-dt}dt.$$

The variance stable carbon,

$$\mathrm{Var}[S(T)] = \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M}|l_m|^2\mathrm{Var}\left[\sum_{j=1}^{N_m(T)}M(t_{j-1}^{(m)},t_j^{(m)})+N(T,t_{N_m(T)})\right], \qquad (2.42)$$

where

$$M(t_{j-1}^{(m)},t_j^{(m)}) = \frac{\eta}{\gamma}\left(X(t_{j-1}^{(m)},t_j^{(m)})+W(t_{j-1}^{(m)},t_j^{(m)})\right),$$

and

$$N(T, t_{N_m(T)}) = \frac{\eta}{\gamma} Y(T, t_{N_m(T)}).$$

Consequently, the variance landscape carbon,

$$\mathrm{Var}[C(T)] \quad = \quad \frac{1}{T^2}\frac{1}{L^2}\sum_{m=1}^{M} |l_m|^2 \mathrm{Var}\left[\sum_{j=1}^{N_m(T)} T(t_{j-1}^{(m)}, t_j^{(m)}) + Z(T, t_{N_m(T)})\right], \quad (2.43)$$

where

$$T(t_{j-1}^{(m)}, t_j^{(m)}) = P(t_{j-1}^{(m)}, t_j^{(m)}) + X(t_{j-1}^{(m)}, t_j^{(m)}) + W(t_{j-1}^{(m)}, t_j^{(m)}) + J(t_{j-1}^{(m)}, t_j^{(m)}) + M(t_{j-1}^{(m)}, t_j^{(m)}),$$

and

$$Z(T, t_{N_m(T)}) = Y(T, t_{N_m(T)}) + N(T, t_{N_m(T)}).$$

2.4. Methods

2.4.1 Parameterizations of the Mathematical Model

For numerical analysis on the behavior of the analytical model we only consider the case of a single stand. To examine the behavior of this model in a real world situation, estimates of 12 parameters are required (Table 1). Although ideally these parameters could be numerically estimated from forest field data, this would take considerable time and effort. To some degree simulation models can mimic the behavior observed in the field. It is therefore logical to use a simulation model to parameterize our analytical formulation for a preliminary check on its ability to predict a field situation. At a minimum this would allow us to test if our analytical solution can predict as well as a simulation model.

In our study we used the Maxcarb model to create data from which to estimate the analytical model parameters. The Maxcarb model calculates temporal changes in carbon storage in seven live pools, six dead pools, three stable pools, and one forest product pool on an annual time step (Smithwick et al in press). The parameters used to predict fluxes between carbon pools in Maxcarb can either be constant over time or vary over time. The latter case reflects the fact that changes in life-forms and species may alter parameter values for processes such as the respiration, mortality, and decomposition. Maxcarb predicts the mass of each pool for each time step using a difference model approach. The parameters used in Maxcarb reflect the general forest stands dynamics through time in the Pacific Northwest (U.S.A.) (Smithwick et al. in press). It should be pointed out that with rare exceptions; the parameters used in Maxcarb are not the same as those used by our analytical model. There is, for example, no maximum amount of live or dead accumulating carbon parameter in Maxcarb.

To create the database from which to calculate our analytical solution parameters, we used Maxcarb to calculate pool sizes for a single stand over 2000 years with one disturbance event at the 1000^{th} year. A thousand year interval was used to make sure all carbon pools in Maxcarb had reached steady-state. We then summed all the live pools together, all the dead pools together, all the stable pools together to produce a total live pool, total dead pool, and total stable pool, respectively. Given that there is only one forest product pool, we used these data directly. Maxcarb does not treat the dead accumulating pool and dead decreasing pool separately. However, we could solve the analytical parameters for the dead decreasing pool and dead accumulating pool because in Maxcarb the dead decreasing pool only receives input and contains stores when a disturbance event has occurred. Thus, in our Maxcarb simulation, the total dead pool over the first thousand year was equal to the mass of only the dead accumulating pool. In contrast, in the second 1000 years, which started with a disturbance, the total dead pool was comprised of both the dead accumulating and the dead decreasing pools. By subtracting the total dead stores of the first 1000 years from that of the second 1000 years, we were able to solve for the mass of the dead decreasing pool by difference. For example, the mass data in the 1^{st} year of the dead decreasing pool is the mass data at the 1001^{st} year subtracted from the mass data in at the 1st year in the total dead pool. Similarly, the mass in the 2^{nd} year of the dead decreasing pool is the mass at the 1002^{nd} year subtracted from the mass in at the 2^{nd} year in the total dead pool. The mass of dead decreasing pool of the remaining 997 years are obtained in a similar way.

Nonlinear regression in Statgraphics Plus was used to fit the mass data for both constant and time varying Maxcarb parameters of the total live pool, the dead accumulating pool, the dead decreasing pool, and the forest product pool, to the functions that model the live accumulating pool, the dead accumulating pool, the dead decreasing pool, and the forest product, respectively and the analytical parameters for these pools were extracted from this analysis (Figures 2 and 3).

The numerical values for the analytical parameters of the stable pool and the transfer rates into the dead decreasing and forest product pools were derived directly from the data set. The parameters of the stable pool were derived in terms of the total input and output masses of the stable pool as well as the total output mass of the dead pool. The total input or output mass of the pools was the combined mass of all the second 1000 years data set. The second 1000 years data set was used so that the mass of both the accumulating dead and decreasing dead pools were included. The stable formation rate was defined to be equal to the total input mass of the stable pool divided by the total output mass of the dead pool and the stable decomposition rate was defined to be equal to the total input mass of the stable pool divided by the total output mass of the stable pool. The transfer rates into the dead decreasing and forest product pools in case of harvest were derived from the data set at the year of the disturbance event occurred and the following year, i.e., the 1000^{th} year and the 1001^{st} year. The transfer rate into the dead decreasing pool was defined to be equal to the output mass of the dead decreasing pool at the 1001^{st} year divided by the combined output mass of the total live and dead accumulating pool at the 1000^{th} year. Similarly, the transfer rate into the forest product pool was defined to be equal to the output mass of the forest product pool at the 1001^{st} year divided by the combined output mass of the total live and dead accumulating pool at the 1000^{th} year.

2.4.2 Behaviors of Analytical Model among Different Disturbance Regimes

Both numerical values of the time constant and time varying Maxcarb parameters (Table 2) were used in examining the behaviors of the analytical model among different disturbance regimes with the exception that the maximum live accumulating carbon was rescaled to one. This put all other pools in terms of their fractional size relative to the live accumulating pool. We used Maple, a mathematical plotting software, to examine both qualitative and quantitative response of the analytical solution to different disturbance regimes. For the comparison of the analytical model among different disturbance regimes, we used Maple to plot the mean landscape of each pool against the mean disturbance interval ranging from 0 to 1,500 years with an infinite observation window. In addition, Maple was used to plot the mean and variance of landscape of each pool versus the mean disturbance interval and/or the observation window in the case random disturbance regime. The observation window is essentially the time the landscape is observed and is related to the size of the landscape. A longer observation window means that a landscape has more stands and/or is being observed for a longer period.

2.4.3 Comparison between Analytical Model and Maxcarb Model

To compare our analytical predictions to a simulation model, we used the MaxCarb model to simulate a landscape under different mean disturbance intervals. While this simulation model was used to produce the parameterization data set, only one disturbance interval (regular intervals of 1000 years) was used. Moreover, the parameters used by the two models are different, with some minor exceptions. Therefore the ability of the Maxcarb model to predict the response to varying types and intervals of disturbance is relatively independent and allows for some degree of consistency that using another model (possibly with different pools) would not. To scale from stand carbon to landscape carbon we used similar method as that of Smithwick et al. (in press). We used the Maxcarb model to simulate a harvest of a single stand for a time period of 30,000 years under a random disturbance regime (with mean disturbance intervals of 25, 50, 75, 100, 125, 150, 175, 200, 250, 300, 350, 450, 500, 750, 1000) for the situation where Maxcarb parameters were either constants over time or varied with time. The later produces much more complex temporal patterns for the carbon pools that may not be well matched by the linear functions assumed in our analytical solution. The 30,000 years period was used to allow as many of the possible stand age classes as possible to be present in the simulated landscape. To calculate the mean landscape carbon we averaged the stand carbon over this 30,000 years period. This time period is equivalent to the observation widow in our analytical model. Finally, we compared the simulated results with the results predicted by the analytical model using linear regression in Excel.

2.5. Results

2.5.1 Analytical Model Parametrization

The numerical values of the analytical parameters estimated using nonlinear regression for both constant and time varying Maxcarb parameters are summarized in Table 2. In general, for both the constant and time varying Maxcarb parameters, the parameters for the analytical model were well estimated, having R^2s above 0.99 (Figure 2). The exception was for the live and dead accumulating pools when the Maxcarb parameters varied with time, which had R^2 as low as 0.92 (Figure 3). The inability of the analytical model functions to closely match Maxcarb in this situation was anticipated given that this Maxcarb parameterization produced non-linear time trends in the pool sizes.

2.5.2 Analytical Model Predictions among Different Disturbance Regimes

While both numerical values of the time constant and time varying Maxcarb parameters were used in examining the behaviors of the analytical model among different disturbance regimes, the two produced the same trends and hence, we only present the results predicted by the time constant Maxcarb parameters.

With no disturbance, the landscape mean for the live accumulating and dead accumulating carbon pools equaled the maximum carbon for these two pools (Figure 4). Under regular and random disturbance regimes the landscape mean live and dead accumulating pools first increased rapidly as the mean of disturbance interval increased and then asymptotically converged to the value without disturbance as the mean disturbance interval increases above 1200 years. The landscape mean live and dead accumulating pool under the random disturbance regime was always greater than that of the regular disturbance regime. Given the parameters we used, the dead accumulating pool stored an equivalent of about 40% of the carbon of the accumulating live pool.

Since disturbance creates the input to the dead decreasing and forest product pools, the landscape means were zero with no disturbance. In both regular and random disturbance regimes, the landscape mean of the dead decreasing and forest product pools decreased as the mean disturbance interval increased, approaching zero as the mean disturbance interval increased to infinity (Figure 5). In contrast to the accumulating live carbon, the landscape mean of the dead decreasing carbon under random disturbance regime was always less than that of the regular disturbance regime. The maximum amounts of carbon in the dead decreasing pool and the forest pool were about 16% and 21% of the accumulating pool, respectively. For both of these the maximum is very dependent on the transfer rates used. In this case, only 72% of the total accumulating carbon transferred to the dead decreasing pool and forest product pool, the remaining 28% was removed by the disturbance. Although the transfer rate to the dead decreasing pool was higher (49% of the accumulating pools) than that of the forest pool(23.4%), the dead decreasing pool had a much faster decomposition rate (0.058724 $year^{-1}$) than that of the forest product pool (0.019795 $year^{-1}$) and hence, the forest products pool retained more carbon on average than the decreasing dead pool.

Similar to the landscape mean live and dead accumulating pools, the landscape mean stable pool increased as the mean disturbance increased and asymptotically leveled off when the mean disturbance interval exceeded 1,000 years (Figure 6). The landscape mean stable pool in the random disturbance regime was always greater than that of the regular disturbance regime. Given our parameters, the stable pool stores an equivalent to about 60% of the carbon in the live accumulating pool.

As expected, given the convex shape of many of the functions used to predict changes in carbon pools, the landscape mean total carbon under both regular and random disturbance regimes was less than that of a landscape without disturbance (at least as long as the mean disturbance interval is less than 1400 years (Figure 6). Similar to the landscape mean accumulating carbon pools, the landscape mean total carbon increased as the mean disturbance interval increased, asymptotically converging to that of the no disturbance landscape as the mean disturbance interval approached infinity. The increase in the landscape mean total as the mean disturbance interval increases is due to the fact that the accumulating pools (live and dead) sequester more carbon than the decreasing pools (dead and forest products). As with the accumulating pools, the landscape mean total carbon under a random disturbance regime is greater than that of the regular disturbance regime.

2.5.3 Effects of Observation Window on Landscape Mean Carbon Stores

Given that our analytical calculations started without carbon stores, the time a landscape is observed influences the amount of carbon estimated in the landscape. For a random disturbance regime, the landscape mean live and dead accumulating pools increased as the observation window increased and asymptotically leveled off as the observation window exceeded the mean disturbance interval (Figure 7). For a given observation window, the mean live and dead accumulating pools also increased as the mean disturbance increased. This is similar to the results in Figure 4.

For a given mean disturbance interval, the mean dead decreasing and forest product pools under random disturbance regime also increased as the observation window increased and asymptotically leveled off as the observation window exceeded 3000 years (Figure 8). This behavior is similar to the accumulating pools because we started our calculations with zero dead stores. This is expected because as the observation window increases more disturbance events occur and there are more carbon inputs into the dead decreasing and forest product pools. For any given observation window, the landscape mean dead decreasing and forest product pools decreased as the mean disturbance interval increased.

The landscape mean stable pool behaved similarly to the live and dead accumulating pool (Figure 9). Similar to the other carbon pools, the landscape mean total carbon increases as the observation window increases. Moreover, it also increased as the mean disturbance interval increased. This is expected given the landscape mean total carbon is comprised of a combination of the other pools which all showed increases as the observation time increased.

2.5.4 Effects of Observation Window and Mean Disturbance Interval on Landscape Variance of Carbon Stores

Without disturbance as well as for a regulated disturbance regime, the variance at the landscape level of each pool as well as the total pool was zero. This is not so for the random disturbance regime. The landscape variance of the live accumulating carbon decreased as the observation widow increased and increased as the mean disturbance interval increased (Figures 12). Similar to the landscape variance of the live accumulating pool, the variance of the landscape dead accumulating carbon decreased as the observation widow increased and increased as the mean disturbance interval increased. Given the parameters we used, the landscape variance of the dead decreasing pool increased when the mean disturbance interval ranged between 0 to 200 years. However, when the mean disturbance interval exceeded 200 years, the landscape variance of the dead decreasing pool decreased (Figure 13). Regarding the observation widow and the mean disturbance interval, the landscape variance of the dead accumulating pool and forest product pools behave exactly as that of the live accumulating pool and dead decreasing pool, respectively, except that they have different numerical values reflecting their different set of parameters. Both the landscape variance of the stable pool and the total pool were similar to the live accumulating pool as the observation widow increased (Figure 14). Also as with the other pools, as the observation window increased toward infinity, the variance of both pools approached zero. Given that the stable pool and the total landscape pool have larger accumulating pools than decreasing pools, the variance of both pools increases as the mean disturbance increased.

2.5.5 Comparison between simulation and analytical approaches

In general, the analytical model predicted similar relationships and trends as those predicted by the MaxCarb model for both when Maxcarb parameters were constant or varied with time. Both the simulation and analytical models predicted that the mean total live carbon, the mean total dead carbon, the total stable carbon, and the total carbon would increase as the mean disturbance interval increased and then asymptotically level off as the mean disturbance interval approached infinity (Figures 15, 16, and 17). The numerical values produced by the analytical model and the MaxCarb model for the live carbon, the dead carbon, and the total carbon are highly correlated, with R^2s above 0.98 (Figures 19 and 20). The analytical model prediction of stable carbon tended to be lower than that of the MaxCarb model, especially when the mean disturbance interval ranged between 100 and 400 years. The greatest difference, about 20%, occurred when the mean disturbance interval was approximately 200 years. However, when the mean disturbance interval was less than 100 or greater than 400 years, the two models matched as well as for the other pools.

2.6. Discussion

The most important result of the work presented here is the incorporation of different disturbance (regular, random, and no disturbance) regimes into an analytical model of long term landscape carbon dynamics. We were able to derive the mean and variance of the landscape carbon under the three different types of disturbance regime. The mean landscape carbon in the case of no disturbance or regular disturbance regimes were straightforward to derive. The variances of landscape carbon under both of these disturbance regimes are zero by definition. The mean and variance of landscape carbon for random disturbance regime were difficult to derive given the assumption of random arrival times between successive disturbances. The analytical solution was derived for arbitrary functional expressions that model carbon pools; however, we only examined those functional expressions modeled by the Maxcarb in our preliminary exploration.

We have reduced a complex problem to a small number of parameters as a first effort to understand how disturbances influence landscape carbon storage. This analytical model represents a first attempt to derive a purely mathematical (versus simulation model) estimate of landscape carbon storage. Instead of including all specific pools as in Maxcarb, we simplified the problem and considered only a few pools that retained the dynamic aspects that are essential to the problem. The analytical model can be modified and expanded to include the same pools as in Maxcarb to reflect a more realistic situation. However, with too many parameters the analytical model can become hard to understand and analyze. Expanding the model to account for additional random variables, for example transfer rates between disturbances, will require new mathematical proof methodologies, as these changes will exceed existing proof capabilities.

The analytical model was aimed at illustrating fundamental principles of how disturbances can influence mean and variance of landscape carbon. Our results showed that the nature of the disturbance regime, which is characterized by both the mean disturbance interval and the observation window, in a landscape can have significant effect on the mean and variance of landscape carbon. Under both regular and random disturbances, the general response of how the mean disturbance interval influences the mean landscape carbon depends highly on the modeling functions which characterize how carbon accumulates or decreases over time in stands. The live and dead accumulating pools were modeled by Chapman-Richards function (which is similar to the complement of a negative exponential function of time), each pool with different set of parameters. The landscape mean of the live and dead accumulating pools as function of the mean disturbance interval exhibited shapes similar to the modeling Chapman-Richards function with asymptotic values approaching the maximum carbons of the live and dead accumulating pools, respectively. However, they both yielded a stronger lag than their corresponding modeling functions because of the averaging effect of concave functions. In contrast, the dead decreasing and forest product pools were modeled by negative exponential functions of time and hence, the landscape mean of the dead decreasing and forest product pools exhibited similar shapes to a negative exponential shape as a function of the mean disturbance interval. In contrast to the mean landscape live and dead accumulating pools, the mean landscape dead decreasing and forest product pools yielded a weaker lag than their corresponding modeling functions because of the averaging effect of convex functions. The landscape mean stable pool was modeled as a proportional combination of the landscape mean of the dead accumulating and dead decreasing pools and hence as a function of the mean disturbance function it exhibited a similar shape to sum of the landscape mean dead accumulating and dead decreasing pools. The landscape mean total carbon were modeled as the sum of all the preceding pools and given the accumulating pools were larger than the decreasing pools, it exhibited a similar shape that of the accumulating pool as a function of the mean disturbance interval.

In general, an increase in total carbon stores as the mean disturbance increased has been predicted in other studies (Kurz et al. 1997-1998; Harmon and Marks 2002; Thornley and Cannell 2004). Thornely and Cannell (2004) showed that reductions in fire intervals from 500 to 100 years halved carbon storage. Smithwick et al. (in press) showed a similar but slightly less serve effect, as the disturbance interval would need to be reduced to from 500 to 25 years to see a 50% reduction in carbon stores. Our analytical prediction is skewed toward that of Smithwick et al.; we predict the mean disturbance interval needs to be reduced from 500 to 30 years to see a 50% reduction in carbon stores. It is obvious that our result is skewed toward that of Smithwick et al. because we both used Maxcarb in our calculations. The asymptotic behavior of the landscape mean total carbon as the mean disturbance interval approaches infinity had also been identified by Smithwick et al. (in press) and can be thought of as the physiological limits of the ecosystem type.

Our analytical model predicted that the total landscape mean carbon under random disturbance regime was always greater than or equal to that of the regular disturbance regime. Smithwick et al. (in press) also found that total landscape carbon storage was higher for the random compared to regular disturbance interval and their explanation was that because random disturbance regimes modeled with a Poisson frequency distribution allow stands that are older than the average disturbance interval to persist on the landscape. We reconfirm and expand that this effect always happens with any distribution of random disturbance intervals (not just with Poisson distribution) as long as the total landscape carbon pool contains significantly more accumulating pools than decreasing pools. Our analytical model predicted that the landscape mean of the live and dead accumulating pools under random disturbance regime was always greater than or equal to that of the regular disturbance regime. Conversely, the landscape mean of the dead decreasing and forest product pools under random disturbance regime was always less than or equal to that of the regular disturbance regime. These results are due to the fact that convex functions were used to model the landscape live and dead accumulating pools and concave functions were used to model the landscape dead decreasing and forest product pools. These contrasting results are explained by the general theorem, known as Jensen's inequality (Ross S.), which states that the expected value of a convex function is greater than or equal to the function of the expected value and conversely for a concave function.

Variance of landscape carbon is an indication of uncertainty in mean landscape carbon. Without disturbance the variance of the landscape carbon is zero as all the stands converge on similar steady-state values. The variance of the landscape carbon is also zero for a regular disturbance regime, because the inter-arrival times of disturbance is a deterministic quantity, and once the age-class structure described by the mean disturbance interval is present, the mean stores does not change. This is not so for the case of random disturbance regime where a distribution of the inter-arrival times of disturbance needs to be specified. As the result, both the observation window and the mean disturbance interval can influence the variation of the landscape carbon. The landscape variance of the (live and dead) accumulating pools, (dead and forest product) decreasing pools, and the stable pool decreased as the observation time increased. For a given mean disturbance interval, increasing the observation time allows more disturbances occur which in turn creates more stand age classes to be present in the landscape and hence results in less variation among the pools. As the observation window approaches infinity, a point where all the possible stand histories are present, the variation of live accumulating carbon theoretically approaches zero. In contrast, for a given observation window the landscape variance of the accumulating pools, and the stable pool increased as the mean disturbance interval increased. This is because given a fixed observation window, increasing the mean disturbance interval results in the occurrence of fewer disturbances which in turn creates fewer stand age classes to be present in the landscape and thus, the landscape carbon variance increases. How the variance of the landscape dead decreasing and forest product pool response to the mean disturbance interval depends on both the input rate created by disturbances and the processes removing carbon from these pools. When the mean disturbance interval was between 0 and 200 years the input created by disturbances was increasing at a positive rate which is similar to the situation of the accumulating pools. This leads to an increase in the variance of the landscape dead decreasing and forest product pools. However, when the mean disturbance interval exceeds 200 years, the input created by disturbance remains relatively constant, and thus the variance of the landscape dead decreasing and forest pools decreases. The observation that the variance of the landscape total carbon increased as the mean disturbance interval increased is due to the fact that total carbon pool in our modeled landscape contained more accumulating pools than decreasing pools. The result that increasing in the variance of landscape carbon as the mean disturbance interval increased is in opposition to the results of Smithwick et al. (in press) and Turner et al. (1993). They found that variation in landscape carbon decreased as the mean disturbance increased.

In general, our results showed that the mean landscape of live carbon, dead carbon, stable carbon, and total carbon as a function of the mean disturbance interval predicted by the analytical model matched consistently well with those predicted by Maxcarb. The predictions of the two models matched regardless of whether the Maxcarb parameters were constant or varied over time. The difference between constant parameters and time varying parameters in Maxcarb is significant from year to year; however, the mean landscape carbon, which is the result of averaging over a long period of time, becomes insensitive to these differences. The consistency in predictions of the two models does indicate the potential for the general application of our analytical model.

While this analytical model can not replace detailed disturbance simulation models such as Maxcarb, it can more expeditiously provide the qualitative trends of these models. In addition, many global carbon models include physiological-based controls, but do not include disturbance effects due to the inherently local and small scale of disturbances. Our analytical model can be used to predict mean and variance of landscape carbon stores for the effects of disturbance regimes. Hence, the analytical model can be combined with the results of physiologically-based models to make adjustments for the presence of disturbances in forested landscapes.

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3. CONCLUSION

We developed an analytical model to address the question of how different disturbance regimes (regular, random , and no disturbance) affect the mean and variance of landscape carbon storage in forest ecosystems. Total landscape carbon was divided into pools based on the processes from which they are derived and their temporal dynamics. This led to five pools: 1) a live accumulating pool, 2) a dead accumulating pool, 3) a dead decreasing pool, 4) a forest product pool, and 5) a stable pool. We were able to derive the mean and variance of the landscape carbon under the three different types of disturbance regime. While the analytical solution could accommodate arbitrary functional expressions that model carbon pools, we only examined those functional expressions modeled by the Maxcarb for preliminary exploration. The simulation model Maxcarb was used to provide data from which to estimate analytical parameters. The predictions of the analytical model were compared with those from Maxcarb. While Maxcarb was used to produce the parameterization data set, only one disturbance interval (regular intervals of 1000 years) was used and hence, the ability of the Maxcarb model to predict the response to varying types and intervals of disturbance is relatively independent that of the analytical model. For both the constant and time varying Maxcarb parameters, the parameters for the analytical model were well estimated, having R^2s above 0.99. The exception was for the live and dead accumulating pools when the Maxcarb parameters varied with time, which had R^2 as low as 0.92. Under both regular and random disturbance regimes the analytical model predicted that mean total landscape carbon increased as mean disturbance interval increased and asymptotically converged to the landscape total carbon without disturbance as the mean disturbance approached infinity. Both observation window and mean disturbance interval influenced the variance of total landscape carbon. Under random disturbance regime, the variance of total landscape carbon increased as the mean disturbance increased and decreased as the observation window increased. The mean landscape total carbon under random disturbance regime was always greater than or equal to that of the regular disturbance regime. The results predicted by the analytical solution consistently matched those predicted by Maxcarb, indicating that while this analytical model can not replace detailed disturbance simulation models such as Maxcarb, it can more expeditiously provide the qualitative trends of these models. The solution to the analytical model was derived for arbitrary functional expressions that model carbon pools and hence, the analytical model can be modified and expanded to include the same pools as in Maxcarb or any other models to reflect a more realistic situation.

Disturbance regimes in landscapes are quite complex. Landscapes are affected by multiple disturbances which occur at different spatial and temporal scales (Turner at el. 1993). A landscape's carbon stores are influenced similarly. The framework we have developed predicts estimates of carbon dynamics in a specified set of disturbance regimes and landscape. We have not attempted to capture all the complexities of landscape carbon dynamics. The simplifying assumptions we made include that stands in the landscape do not interact with each other; that each disturbance leave no live vegetation, and that disturbances transferred a constant percentage of carbon from the live and dead accumulating pools to decreasing pools do not describe all the behaviors of a natural forested landscape. Our hope in the future will be to enhance the model by incorporating additional variables that would relax these assumptions.

List of Tables

Analytical parameters	Description
A	maximum live accumulating carbon
a	live accumulation rate
q	lag for live accumulating pool
α	dead fraction of live accumulating pool
b	dead accumulation rate
p	lag for dead accumulating pool
μ	transfer rate into dead decreasing pool, natural harvest or natural
c	decomposition rate for dead material
β	transfer rate into forest product pool natural harvest or natural
d	decomposition rate for forest product
η	stable formation rate
γ	stable decomposition rate

TABLE 1: Analytical parameters

Analytical parameters	Numerical values for constant Maxcarb parameters	Numerical values for time varying Maxcarb paramters	Unit
A	354.077	353.375	Mg/ha
a	0.013554	0.010369	$year^{-1}$
q	1	1	unitless
α	0.407023	0.394895	$year^{-1}$
b	0.013872	0.018757	$year^{-1}$
p	1	1	unitless
μ	0.492055	0.514229	unitless
c	0.058724	0.058070	$year^{-1}$
β	0.234158	0.231459	unitless
d	0.019795	0.019795	$year^{-1}$
η	0.015638	0.012784	$year^{-1}$
γ	0.026719	0.023541	$year^{-1}$

TABLE 2: Numerical values for constant and time varying MaxCarb parameters

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